Math 246C Lecture 14 Notes

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1 Uniformization Case 2 and Green's Functions Away From a Disc

1.1 Uniformization, Case 2 (cont.)

Last time, we were finishing our proof of the Uniformization theorem.

Theorem 1.1 (Uniformization, Case 2)**.** *Let X be a simply connected Riemann surface for which Green's function does not exist. If X is compact, then there is a holomorphic bijection $X \to \hat{\mathbb{C}}$. If X is not compact, there is a holomorphic bijection $X \to \mathbb{C}$.*

Proof. If G_{x_1,x_2} is a dipole Green's function, then there is a $\varphi \in \mathrm{Hol}(X, \hat{\mathbb{C}})$ such that $|\varphi(y)| = e^{-G_{x_1,x_2}(y)}$, $\varphi(x_1) = 0$, and $\varphi(x_2) = \infty$ (a simple pole). We only need to show that φ is injective on X. Let $x_0 \in X \setminus \{x_1, x_2\}$. The dipole Green's function $G_{x_0,x_2}(y)$ exists, then there is a $\varphi_0 \in \mathrm{Hol}(X, \hat{\mathbb{C}})$ such that $|\varphi_0(y)| = e^{-G_{x_0,x_2}(y)}$ for $y \in X$. Consider the function
$$f(y) = \frac{\varphi(y) - \varphi(x_0)}{\varphi_0(y)},$$
which is holomorphic away from x_0, x_2. The singularities at x_0, x_2 are removable, so $f \in \mathrm{Hol}(X)$.

Now
$$\sup_{y \in X \setminus (D_1 \cup D_2)} < \infty \implies |f(y)| \leq e^{G_{x_0,x_2}(y)}(e^{-G_{x_1,x_2}(y)} + C),$$
so f is bounded away from x_0, x_1, x_2. Since f is holomorphic at these 3 points, f is bounded on all of X. Say $|f(y)| \leq M$. Let $v \in \mathcal{F}_{x_1}$ be a Perron amily for G_{x_1}. Then
$$v(y) + \log\left|\frac{f(y) - f(x_1)}{2M}\right|, \qquad y \in X \setminus \{x_1\}$$
by the Lindelöf maximum principle. Since $\sup_{v \in \mathcal{F}_{x_1}} v(y) = \infty$ for all y, we get $f(y) = f(x_1)$ for all $y \in X$.

We get that

$$\frac{\varphi(y)-\varphi(x_0)}{\varphi_0(y)} = \frac{\varphi(x_1)-\varphi(x_0)}{\varphi_0(x_1)} = -\frac{\varphi(x_0)}{\varphi_0(x_1)} \notin \{0,\infty\}.$$

In particular, $\varphi \neq \varphi(x_0)$ unless $\varphi_0(y) = 0$. This is when $y = x_0$. Thus, φ is injective on $X \setminus \{x_1, x_2\}$ and hence on X. $\square$

1.2 Existence of a Green's function away from a disc

It now remains to prove the existence of a dipole Green's function. We need the following fact.

Theorem 1.2. *Let X_0 be a Riemann surface, and let $D_0 \subseteq X_0$ be a parametric disc. Set $X = X_0 \setminus \overline{D}_0$. Then for all $x \in X$, a Green's function $G_x(y)$ on X exists.*

Given this construction, we can produce a dipole Green's function by taking the difference of Green's functions G_{x_1} and G_{x_2} for $x_1, x_2 \notin \overline{D}_0$. Then we can shrink the size of the disc to try to get a dipole Green's function on all of X_0.

Proof. Let $x \in X$, and let $S \subseteq X$ be a parametric disc $D \subseteq X$ with $x \in D \cong \{|z| < 1\}$ and $z(x) = 0$. When $0 < r < 1$, let $rD = \{y \in D : |z(y)| < r\}$. Let $v \in \mathcal{F}_x$, a Perron family on X. Then

$$v(y) + \log|z(y)| \leq \sup_{\partial D} V, \qquad y \in D, y \neq x$$

by the Lindelöf maximum principle. In particular,

$$\sup_{y \in \partial(rD)} v(y) + \log(r) \leq \sup_{\partial D} v.$$

Idea: We want to solve the Dirichlet problem[1] on $X \setminus \overline{rD} = X_0 \setminus (\overline{D}_0 \cup \overline{rD})$:

$$\Delta u = 0 \text{ on } X \setminus \overline{rD}, \qquad u|_{\partial(rD)} = 1, \qquad u|_{\partial D_0} = 0.$$

We will use Perron's method. Let $\mathcal{F}$ be the collection of us which are subharmonic on $X \setminus \overline{rD}$, $u = 0$ far away, and such that

$$\limsup_{y \to \zeta} u(y) \leq 1 \qquad \forall \zeta \in \partial(rD),$$

$$\limsup_{y \to \alpha} u(y) \leq 0 \qquad \forall \alpha \in \partial D_0.$$

[1]We have not formally defined the Laplacian on a Riemann surface, but this should at least motivate the rest of the proof.

For all $u \in \mathcal{F}$, $u \leq 1$, so by the Perron theorem,

$$\omega(y) = \sup_{v \in \mathcal{F}} v(y)$$

is harmonic on $X \setminus \overline{rD}$.

Any point $\xi \in \partial D_0 \cup \partial(rD)$ is a **regular point** for the Dirichlet problem in the sense that there is a local barrier at ξ: Recall that h is a **local barrier** at $\xi \in \partial\Omega$ (where $\Omega \subseteq \mathbb{C}$ is open and connected) if

1. h is defined and subharmonic on $\Omega \cap V$ for some neighborhood B of ξ.
2. $h(z) < 0$ in $\Omega \cap V$
3. For $z \in \Omega$ $h(z) \to 0$ as $z \to \xi$.

If $\partial\Omega \in C^1$, then any $\xi \in \partial\Omega$ is a regular point. By Perron's theorem, it follows that $\omega = \sup v$ extends continuously to $\partial(rD) \cup \partial D_0$. So we have a harmonic ω on $X \setminus \overline{rD}$ such that $\omega|_{\partial(rD)} = 1$ and $\omega|_{\partial D_0} = 0$. We have that $0 \leq \omega \leq 1$, and by the maximum principle, $0 < \omega < 1$ on $X \setminus \overline{rD}$.

Let us go back to $v \in \mathcal{F}_x$:

$$\sup_{y \in \partial(rD)} v(y) + \log(r) \leq \sup_{\partial D} v.$$

Consider the subharmonic function on $X \setminus \overline{rD}$

$$v - \left(\sup_{\partial(rD)} v \right) \omega.$$

By the maximum principle, this function is ≤ 0. So

$$v \leq \left(\sup_{\partial D} v \right) w,$$

which gives us that

$$\sup_{\partial D} v \leq \left(\sup_{\partial(rD)} v \right) \underbrace{\sup_{\partial D} \omega}_{=1-\delta}.$$

Combining this with our previous bound on v gives

$$\delta \sup_{\partial(rD)} \leq \sup_{\partial(rD)} v - \sup_{\partial D} v,$$

so

$$\delta \sup_{\partial(rD)} + \log(r) \leq 0.$$

We get that

$$\sup_{\partial(rD)} \leq \frac{1}{\delta} \log\left(\frac{1}{r}\right), \qquad \forall v \in \mathcal{F}_x$$

Thus, $\sup_{v\in\mathcal{F}} v \neq \infty$, and G_x exists. $\square$

Remark 1.1. The function ω is called the **harmonic measure** of $\partial(rD)$ in the region $X \setminus \overline{rD}$.